

# Shear viscosity of an ultrarelativistic Boltzmann gas with isotropic inelastic scattering processes.

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## Abstract

We derive an analytic expression for the shear viscosity of an ultra-relativistic gas in presence of both elastic  $2 \rightarrow 2$  and inelastic  $2 \leftrightarrow 3$  processes with isotropic differential cross sections. The derivation is based on the entropy principle and Grad's approximation for the off-equilibrium distribution function. The obtained formula relates the shear viscosity coefficient  $\eta$  to the total cross sections  $\sigma_{22}$  and  $\sigma_{23}$  of the elastic resp. inelastic processes. The values of shear viscosity extracted using the Green-Kubo formula from kinetic transport calculations are shown to be in excellent agreement with the analytic results which demonstrates the validity of the derived formula.

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## I. INTRODUCTION

Dynamics of the hot and dense state of nuclear matter, produced in heavy-ion collisions at modern accelerator facilities like RHIC at Brookhaven National Laboratory or LHC at CERN, can be described by means of kinetic transport theory or relativistic hydrodynamics. In fact, these two approaches have been widely used for research of the properties of the so-called Quark-Gluon Plasma (QGP), early-time dynamics of which determines the properties of hadronic and electromagnetic secondaries measured experimentally [1–3]. Whereas the kinetic transport theory is based on a microscopic paradigm and describes evolution of the phase-space distribution function by solving the corresponding transport equations, hydrodynamics is a macroscopic theory describing space and time evolution of macroscopic fields like the energy-momentum tensor and particle number four-vector. For the hydrodynamic theory the transport coefficients are an extrinsic input and must be calculated from the underlying field theory, which for QGP is the Quantum Chromodynamics (QCD). Calculations of the transport coefficients are based on the formal correspondence between the kinetic theory and the hydrodynamics, which can be obtained by integration of the microscopic equations over the phase space. For example, such correspondence is discussed in [4–8]. Moreover, kinetic transport calculations have often been used as a benchmark for hydrodynamic models, as for instance in Refs. [9–11], which again requires a consistent connection between the cross section on the one hand and transport coefficients on the other.

For a strongly interacting QGP, the shear viscosity coefficient has been calculated in the perturbative regime as function of the coupling parameter  $\alpha_s$  for instance in Refs. [12], [13–15]. Such calculations account for the radiative processes occurring in the QGP, which have an important contribution to the isotropization of the medium. The role of inelastic processes was understood by means of the transport cross section or transport rate [18], which are analogous to the total cross section and collision rate but are additionally weighted by the angle of outgoing particles. In contrast, calculations reported in Refs. [16] do not see a strong contributions of the inelastic  $2 \rightarrow 3$  processes. In Ref. [17] the shear viscosity of a QGP was calculated employing two-particle scatterings, i.e. without taking into account the radiative processes.

In this work we derive an expression which relates the shear viscosity coefficient of an ultrarelativistic gas to the total elastic and inelastic cross sections in the medium. For our

derivation we consider isotropic, i.e. momentum-independent, matrix elements for the collisional processes, thus bypassing the conceptual difficulties connected with the momentum-dependent matrix elements in a (perturbative) QCD medium. To our best knowledge, analytic derivation of the shear viscosity for  $2 \leftrightarrow 3$  inelastic processes with isotropic differential cross sections is completely new.

This paper has the following structure. In Section II we start with the derivation of an expression for the shear viscosity coefficient, for which we use the entropy production in a system off equilibrium. In order to obtain a closed analytic expression for the shear viscosity coefficient in terms of the involved total cross sections, kinematic integrals will be evaluated. This is discussed in Section III, where the final expression is also presented. In Section IV the results of the analytic formula are compared with Green-Kubo based kinetic transport calculations of the shear viscosity coefficient. Finally, Section V contains summary and conclusions.

## II. DERIVATION OF THE EXPRESSION FOR THE SHEAR VISCOSITY COEFFICIENT

The derivation we follow is built upon the entropy principle, i.e. the second law of thermodynamics is used. In presence of dissipative fields, of which we will consider only the shear stress tensor throughout this work, the entropy production  $\partial_\mu s^\mu$  must have a non-negative form, and hence can be written as discussed e.g. in Ref. [19]:

$$\partial_\mu s^\mu = \frac{\pi_{\mu\nu}\pi^{\mu\nu}}{2\eta T} + J \ln \lambda \geq 0. \quad (1)$$

In the latter expression  $\pi^{\mu\nu}$  denotes the shear tensor and  $\eta$  the shear viscosity coefficient;  $T$  is the temperature and  $\lambda$  the fugacity, which are both related to the chemical potential,  $\ln \lambda \equiv \frac{\mu}{T}$ .  $J$  denotes a source for particle production and annihilation and vanishes in the situation where the net particle number does not change.

Note that  $\eta$  in Eq. (1) is identified with the shear viscosity since one can show that in the lowest-order, i.e. Navier-Stokes, dissipative hydrodynamic theory, where  $\pi^{\mu\nu} = 2\eta \nabla^{\langle\mu} u^{\nu\rangle}$ , the entropy production indeed takes the form (1). Since higher-order theories (such as Israel-Stewart [4]) are supposed to converge to the Navier-Stokes form, the entropy production is always written in the form (1) and a consistent dynamic equation for  $\pi^{\mu\nu}$  can be found, like

e.g. in Ref. [5].

On the other hand, the entropy production can be obtained from the total divergence of the entropy current defined in the kinetic theory as

$$s^\mu = - \int p^\mu f(x, p) [\ln f(x, p) - 1] d\Gamma, \quad (2)$$

where  $f(x, p)$  denotes the phase-space distribution function and  $d\Gamma \equiv d^3p/(2\pi)^3/E$ . For systems away from kinetic equilibrium the distribution can be written introducing a small deviation  $\phi(x, p)$  from the equilibrium form  $f_0(x, p)$ :

$$f(x, p) = f_0(x, p) \cdot (1 + \phi(x, p)). \quad (3)$$

Some constraints must be applied to Eq. (3) in order to preserve consistency. For instance, if moments of  $f(x, p)$  from (3) are calculated, the particle number four-vector  $N^\mu$  and the energy-momentum tensor  $T^{\mu\nu}$  must be recovered. This means that the deviation  $\phi(x, p)$  must depend on the deviations  $\delta T^{\mu\nu}$  and  $\delta N^\mu$  of the energy-momentum tensor resp. particle number four-vector from their ideal forms. The explicit functional dependence is given by the following expression, as obtained in scope of Grad's method of moments [13, 20–22]:

$$\phi(x, p) = c(e, p, T) \cdot \pi_{\mu\nu} p^\mu p^\nu. \quad (4)$$

Note that the momentum-dependence of  $\phi$  can be extended beyond the second power, as demonstrated in Ref. [24]. However, our derivation is restricted to the original approach of Grad, for which the series must be truncated at second power of momentum. In the kinetic theory, dynamics of  $f(x, p)$  is governed by a transport equation, such as the Boltzmann Equation

$$p^\mu \partial_\mu f(x, p) = \mathcal{C}[f(x, p)], \quad (5)$$

with the functional  $\mathcal{C}[f(x, p)]$  on the right hand side denoting the collision term. Combining Eqs. (2), (3), (5) and (1) one obtains the following expression for the shear viscosity coefficient [13, 22]

$$\eta = \frac{\pi_{\mu\nu} \pi^{\mu\nu}}{2c(e, p, T) \cdot T \pi_{\mu\nu} \mathcal{P}^{\mu\nu}}, \quad (6)$$

with  $\mathcal{P}^{\mu\nu}$  denoting the second moment of the collision term

$$\mathcal{P}^{\mu\nu} = \int p^\mu p^\nu \mathcal{C}[f] d\Gamma. \quad (7)$$

An obvious property of the obtained expression is the explicit dependence on the shear tensor components. But since the shear viscosity coefficient is a medium property and should not depend on the dynamics encoded in  $\pi^{\mu\nu}$ , Eq. (6) should be considered in the limit  $\pi^{\mu\nu} \rightarrow 0$ .

The functional  $\mathcal{P}^{\mu\nu}$  contains information about the collisional processes in the medium. We consider only local interactions between constituents of the medium and assume them to be Boltzmann-particles, i.e. quantum statistics effects are neglected. If both elastic and inelastic interactions are considered, the second moment of the collision term takes the following form:

$$\begin{aligned}
\mathcal{P}^{\mu\nu} = & \int d\Gamma p^\mu p^\nu \mathcal{C} = \\
& \frac{1}{2} \int dw_1 dw_2 \frac{1}{2} \int dw'_1 dw'_2 p_1^\mu p_1^\nu f'_1 f'_2 |M_{1'2' \rightarrow 12}|^2 (2\pi)^4 \delta^{(4)}(p'_1 + p'_2 - p_1 - p_2) - \\
& \frac{1}{2} \int dw_1 dw_2 \frac{1}{2} \int dw'_1 dw'_2 p_1^\mu p_1^\nu f_1 f_2 |M_{12 \rightarrow 1'2'}|^2 (2\pi)^4 \delta^{(4)}(p'_1 + p'_2 - p_1 - p_2) + \\
& \frac{1}{2} \int dw_1 dw_2 dw_3 \frac{1}{2} \int dw'_1 dw'_2 p_1^\mu p_1^\nu f'_1 f'_2 |M_{1'2' \rightarrow 123}|^2 (2\pi)^4 \delta^{(4)}(p'_1 + p'_2 - p_1 - p_2 - p_3) + \\
& + \int dw_1 dw_2 \frac{1}{6} \int dw'_1 dw'_2 dw'_3 p_1^\mu p_1^\nu f'_1 f'_2 f'_3 |M_{1'2'3' \rightarrow 12}|^2 (2\pi)^4 \delta^{(4)}(p'_1 + p'_2 + p'_3 - p_1 - p_2) - \\
& - \frac{1}{2} \int dw_1 dw_2 dw_3 \frac{1}{2} \int dw'_1 dw'_2 p_1^\mu p_1^\nu f_1 f_2 f_3 |M_{123 \rightarrow 1'2'}|^2 (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 - p'_1 - p'_2) - \\
& - \int dw_1 dw_2 \frac{1}{6} \int dw'_1 dw'_2 dw'_3 p_1^\mu p_1^\nu f_1 f_2 |M_{12 \rightarrow 1'2'3'}|^2 (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p'_1 - p'_2 - p'_3)
\end{aligned} \tag{8}$$

with  $dw \equiv \frac{1}{2} d\Gamma = \frac{1}{2(2\pi)^3} \frac{d^3 p}{E}$ . The first two terms in (8) account for elastic  $2 \rightarrow 2$  processes, whereas the last four terms capture the inelastic  $2 \rightarrow 3$  and reverse  $3 \rightarrow 2$  processes. The numerical factors  $1/2$  and  $1/6$  account for multiple counting of identical Boltzmann particles.

In order to evaluate the integrals in (8) analytically, the distribution functions will be replaced by the near-equilibrium approximations (3) and only terms up to first order in  $\phi$  will be considered:

$$\prod_i f_i \approx \left( 1 + \sum_j \phi_j \right) \cdot \prod_i f_{0,i}. \tag{9}$$

The equilibrium distribution  $f_0$  is the Boltzmann distribution

$$f_0 = d \cdot e^{-p_\alpha u^\alpha / T + \mu / T}, \tag{10}$$

with the degeneracy factor  $d$ .

Analytic evaluation of the integrals in (8) can be simplified if a specific symmetry for the geometry of the system is considered. Since the shear viscosity coefficient is a material property, its value cannot depend on the form of the energy-momentum tensor distortion, i.e. any plausible form for the shear tensor can be considered to evaluate (8). The simplest geometry that can be considered for the shear tensor, which has to be traceless, is a diagonal form with one independent component  $\pi$ :

$$\pi_{\mu\nu} = \text{diag}(0, \pi/2, \pi/2, -\pi). \quad (11)$$

This form of the shear tensor implies deviation from isotropy in  $z$ -direction and isotropy in the transverse plane. Systems with this kind of symmetry have been studied e.g. in Refs. [11, 19]. With the shear tensor given by Eq. (11), the expression for the shear viscosity (6) as well as the collision term moment (8) are significantly simplified, since now  $\pi$  is the only unknown parameter, for which the  $\pi \rightarrow 0$  limit must be considered. In fact, with the introduced form of the shear pressure tensor the correction (4) to the distribution function becomes (for sake of compactness we will suppress the dependence of  $c(\cdot)$  on  $e, p, T$  in the notation)

$$\phi(x, p) = c \cdot \pi \cdot \left( \frac{1}{2} p_T^2 - p_z^2 \right). \quad (12)$$

Now the contraction  $\pi_{\mu\nu} \mathcal{P}^{\mu\nu}$  in (6) can be rewritten as follows

$$\begin{aligned} \pi_{\mu\nu} \mathcal{P}^{\mu\nu} &= \pi_{\mu\nu} \int d\Gamma p^\mu p^\nu \mathcal{C} = \\ &\frac{\pi}{4} \int dw_1 dw_2 dw'_1 dw'_2 \left( \frac{1}{2} p_{1,T}^2 - p_{1,z}^2 \right) \cdot f'_{0,1} f'_{0,2} \times \\ &\times \left( 1 + c \cdot \pi \cdot \left( \frac{1}{2} p_{1,T}^2 - p_{1,z}^2 \right) + c \cdot \pi \cdot \left( \frac{1}{2} p_{2,T}^2 - p_{2,z}^2 \right) \right) \cdot |M_{1'2' \rightarrow 12}|^2 (2\pi)^4 \delta^{(4)}(..) - \\ &\frac{\pi}{4} \int dw_1 dw_2 dw'_1 dw'_2 \left( \frac{1}{2} p_{1,T}^2 - p_{1,z}^2 \right) \cdot f_{0,1} f_{0,2} \times \\ &\times \left( 1 + c \cdot \pi \cdot \left( \frac{1}{2} p_{1,T}^2 - p_{1,z}^2 \right) + c \cdot \pi \cdot \left( \frac{1}{2} p_{2,T}^2 - p_{2,z}^2 \right) \right) \cdot |M_{12 \rightarrow 1'2'}|^2 (2\pi)^4 \delta^{(4)}(..) + \\ &\frac{\pi}{4} \int dw_1 dw_2 dw_3 dw'_1 dw'_2 \left( \frac{1}{2} p_{1,T}^2 - p_{1,z}^2 \right) f'_{0,1} f'_{0,2} \times \\ &\times \left( 1 + c \cdot \pi \cdot \left( \frac{1}{2} p_{1,T}^2 - p_{1,z}^2 \right) + c \cdot \pi \cdot \left( \frac{1}{2} p_{2,T}^2 - p_{2,z}^2 \right) \right) \cdot |M_{1'2' \rightarrow 123}|^2 (2\pi)^4 \delta^{(4)}(..) + \\ &\frac{\pi}{6} \int dw_1 dw_2 dw'_1 dw'_2 dw'_3 \left( \frac{1}{2} p_{1,T}^2 - p_{1,z}^2 \right) \cdot f'_{0,1} f'_{0,2} f'_{0,3} \times \\ &\times \left( 1 + c \cdot \pi \cdot \left( \frac{1}{2} p_{1,T}^2 - p_{1,z}^2 \right) + c \cdot \pi \cdot \left( \frac{1}{2} p_{2,T}^2 - p_{2,z}^2 \right) + c \cdot \pi \cdot \left( \frac{1}{2} p_{3,T}^2 - p_{3,z}^2 \right) \right) \cdot |M_{1'2'3' \rightarrow 12}|^2 (2\pi)^4 \delta^{(4)}(..) - \\ &\frac{\pi}{4} \int dw_1 dw_2 dw_3 dw'_1 dw'_2 \left( \frac{1}{2} p_{1,T}^2 - p_{1,z}^2 \right) \cdot f_{0,1} f_{0,2} f_{0,3} \times \end{aligned}$$

$$\begin{aligned}
& \times \left( 1 + c \cdot \pi \cdot \left( \frac{1}{2} p_{1,T}^2 - p_{1,z}^2 \right) + c \cdot \pi \cdot \left( \frac{1}{2} p_{2,T}^2 - p_{2,z}^2 \right) + c \cdot \pi \cdot \left( \frac{1}{2} p_{3,T}^2 - p_{3,z}^2 \right) \right) \cdot |M_{123 \rightarrow 1'2'}|^2 (2\pi)^4 \delta^{(4)}(..) - \\
& \frac{\pi}{6} \int dw_1 dw_2 dw'_1 dw'_2 dw'_3 \left( \frac{1}{2} p_{1,T}^2 - p_{1,z}^2 \right) \cdot f_{0,1} f_{0,2} \times \\
& \left( 1 + c \cdot \pi \cdot \left( \frac{1}{2} p_{1,T}^2 - p_{1,z}^2 \right) + c \cdot \pi \cdot \left( \frac{1}{2} p_{2,T}^2 - p_{2,z}^2 \right) \right) \cdot |M_{12 \rightarrow 1'2'3'}|^2 (2\pi)^4 \delta^{(4)}(..)
\end{aligned} \tag{13}$$

The parts of the integrals that do not contain  $c$  (i.e. those parts where the integration goes over  $f_0$  and not  $f_0 \cdot \phi$ ) have to vanish, since there is no entropy production in an equilibrated medium. The remaining integrals will then be of order  $\pi^2$ , just as the expression in the numerator of (6), so that the shear viscosity coefficient will not depend on  $\pi$ , as it should be.

### III. EVALUATION OF THE KINEMATIC INTEGRALS

In general the matrix elements  $|M_{n \rightarrow m}|$  will have a highly non-trivial dependence on momenta of the colliding particles and analytic evaluation might be difficult. Here we will consider angle-independent matrix elements to obtain the shear viscosity as function of total cross section. The total cross sections  $\sigma_{22}$  and  $\sigma_{23}$  for elastic resp. inelastic two-particle processes can be defined according to [2, 23]:

$$\sigma_{22} = \frac{1}{2s} \frac{1}{2} \int dw_1 \int dw_2 |M_{1'2' \rightarrow 12}|^2 (2\pi)^4 \delta(p_1 + p_2 - p'_1 - p'_2) , \tag{14}$$

$$\sigma_{23} = \frac{1}{2s} \frac{1}{6} \int dw_1 \int dw_2 \int dw_3 |M_{1'2' \rightarrow 123}|^2 (2\pi)^4 \delta(p_1 + p_2 + p_3 - p'_1 - p'_2) . \tag{15}$$

For these definitions the center-of-mass energy squared of the colliding particles  $s = (p_1 + p_2)^2$  has been introduced. For a process with 3 particles in the initial state a classical cross section cannot be defined, but one can define an analogous quantity [2]

$$\mathcal{I}_{32} = \frac{1}{2} \int dw_1 \int dw_2 |M_{1'2'3' \rightarrow 12}|^2 (2\pi)^4 \delta(p_1 + p_2 - p'_1 - p'_2 - p'_3) . \tag{16}$$

The squared matrix elements  $|M_{2 \rightarrow 3}|^2$  and  $|M_{3 \rightarrow 2}|^2$  depend on each other by virtue of the detailed balance requirement:

$$|M_{2 \rightarrow 3}|^2 = d \cdot |M_{3 \rightarrow 2}|^2 . \tag{17}$$

If the matrix elements are angle-independent, i.e. if the scattering is *isotropic*, as considered for the following derivations, one can obtain simple relations between them, the cross

sections  $\sigma_{22}$ ,  $\sigma_{23}$  and the quantity  $\mathcal{I}_{32}$  [2] by performing the momentum integration in the definitions (14) – (16). Note that the right-hand-sides of Eqs. (14) – (15) are Lorentz-invariant and thus can be conveniently evaluated in the center-of-mass frame of colliding particle pair:

$$|M_{2 \rightarrow 2}|^2 = 32\pi \cdot s \cdot \sigma_{22}, \quad (18)$$

$$|M_{2 \rightarrow 3}|^2 = 192\pi^3 \cdot \sigma_{23}, \quad (19)$$

$$\mathcal{I}_{32} = \frac{192}{d}\pi^2\sigma_{23}. \quad (20)$$

Here, again,  $d$  is the degeneracy factor of the constituent particles.

With the definitions (14) – (16) half of the terms (the second, fifth and sixth terms) in (8) are significantly simplified. Calculation of the remaining terms is a bit more involved. The first, third and fourth terms in (8) contain integrals over the  $n = 2$  or 3 *final state momenta*  $dw_1 \dots dw_n$  which can be formally written as

$$\int dw_1 \dots \int dw_n \mathcal{G}(p_1 \dots p_n) |M_{m \rightarrow n}|^2 \delta\left(\sum_m p_{\text{initial}} - \sum_n p_{\text{final}}\right). \quad (21)$$

In the latter expression  $\mathcal{G}$  denotes a (polynomial) function of the final state momenta. For the matrix elements the relations (18) – (20) can be used. However it is important to note that the expression (21) is not anymore Lorentz-invariant due to presence of the function  $\mathcal{G}$ . At least for the terms describing a two-particle initial state it is still convenient to transform the integral into the center-of-mass frame for evaluation and to apply a back-transform afterwards. This means rewriting Eq. (21) in the following form

$$\int dw_1^* \dots \int dw_n^* \mathcal{G}(p_{TR,1}^* \dots p_{TR,n}^*) |M_{m \rightarrow n}|^2 \delta\left(\sum_m p_{\text{initial}}^* - \sum_n p_{\text{final}}^*\right). \quad (22)$$

where  $*$  denotes quantities in the center-of-mass frame, in which  $E_1^* + E_2^* = \sqrt{s}$  and  $\vec{p}_1^* + \vec{p}_2^* = 0$ . To rewrite (21) into the form (22) Lorentz invariance of  $dw_i$  and of the 4-dimensional Delta-function can be used. Notation  $\mathcal{G}(p_{TR,1}^* \dots p_{TR,n}^*)$  means that function  $\mathcal{G}(\cdot)$  acts on momenta that are Lorentz-transformed from the center-of-mass frame into the lab frame:

$$E_{TR}^* = \gamma(E^* + \vec{p}^* \vec{\beta}_{CM}) \quad (23)$$

$$\vec{p}_{TR}^* = \vec{p}^* + \frac{\vec{\beta}_{CM} \gamma_{CM}^2}{1 + \gamma_{CM}} \vec{\beta}_{CM} \cdot \vec{p}^* + \gamma_{CM} \vec{\beta}_{CM} E^*. \quad (24)$$



with the Lorentz boosts

$$\vec{\beta}_{CM} = \frac{\vec{p}_1 + \vec{p}_2}{E_1 + E_2}, \quad (25)$$

$$\gamma_{CM} = \frac{1}{\sqrt{1 - \vec{\beta}_{CM}^2}}. \quad (26)$$

For the term describing a three-particle initial state (the fourth term in (8) resp. (13)) such transformation into and then back from the center-of-mass frame does not induce any significant simplification of the calculation so that the integration must be evaluated directly in the lab frame.

After evaluating all the integrals we finally obtain the following expression for the shear viscosity coefficient:

$$\eta = \frac{6}{5} \cdot \frac{T}{\sigma_{22} + \frac{6}{5}\sigma_{23} + \frac{3}{10}\lambda\sigma_{23}}. \quad (27)$$

The second and the third terms in the denominator of the latter equation account for contributions of  $2 \rightarrow 3$  resp.  $3 \rightarrow 2$  processes. The factor  $\lambda$  in the  $3 \rightarrow 2$  term denotes the fugacity, which is a measure for the degree of saturation of the particle density in a system

$$\lambda = n/n_{eq}, \quad (28)$$

with  $n_{eq} = d \cdot \int e^{-u_\mu p^\mu/T} d^3p$ . Setting  $\lambda = 1$  (or, accordingly,  $\mu = 0$  in Eq. (10)), i.e. considering a chemically equilibrated system, we obtain

$$\boxed{\eta = \frac{6}{5} \cdot \frac{T}{\sigma_{22} + \frac{3}{2}\sigma_{23}}} \quad (29)$$

Neglecting the contribution from inelastic scatterings, this formula is identical with the already known results from Refs. [21] and [24].

#### IV. COMPARISON WITH KINETIC TRANSPORT CALCULATIONS

In order to confirm the validity of Eq. (29), one can compare it to the values of shear viscosity extracted from the Boltzmann Approach to Multi-Parton Scatterings (BAMPS) [2, 18] using the Green-Kubo formalism. In Ref. [14] we have recently reported, how the Green-Kubo formalism can be applied to extract shear viscosity from BAMPS kinetic transport simulations. In Fig. 1 we demonstrate the shear viscosity values extracted this way from BAMPS calculations for which only inelastic  $2 \rightarrow 3$  and  $3 \rightarrow 2$  processes were

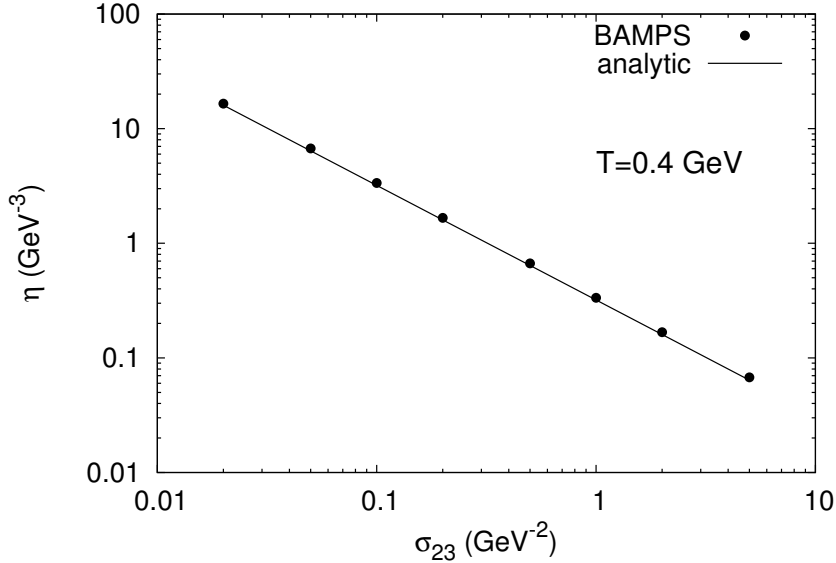


FIG. 1. Shear viscosity in presence of inelastic processes only, extracted from BAMPS using Green-Kubo formalism (symbols) and calculated using Eq. (29). The results are obtained employing angle-independent (isotropic) differential cross sections.

considered. The shear viscosity is shown as function of the inelastic cross section  $\sigma_{23}$ . We observe a very good agreement of the results extracted from BAMPS (symbols) with the results obtained using Eq. (29) (lines) derived in this work. We also would like to mention that Eq. (29) confirms the values of the shear viscosity which we obtained in Ref. [19] by numerical evaluation of the Equation (6). Equation (29) implies that if the total cross sections are equal, relaxation of the medium towards equilibrium will proceed 1.5 times faster if only inelastic processes are involved as compared to the case when only elastic processes are involved (since the hydrodynamic relaxation time is directly proportional to the shear viscosity coefficient – comp. [5, 10]). This reflects the fact that the inelastic processes are more efficient in driving the medium towards equilibrium, as was demonstrated in Refs. [13, 18, 19].

## V. CONCLUSIONS

We have derived an analytic expression relating the shear viscosity coefficient of an ultra-relativistic gas to the total elastic and inelastic cross sections of isotropic scattering processes

in the medium. From the obtained expression one recognizes that, given all total cross sections are equal, isotropic inelastic processes are 1.5 times more efficient than the elastic ones in restoring kinetic equilibrium in a system. The values of the shear viscosity obtained by Eq. (29) are in very good agreement with the results of Green-Kubo based calculations performed using the kinetic transport model BAMPS. The correspondence between shear viscosity and total cross sections of elastic and inelastic processes reported is another step to establish a connection between the kinetic transport and dissipative hydrodynamic models.

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